Latticization of Ordinal Numbers

Abstract

In this paper our endeavour is to build up a theory of ordinal lattices which emerges from the consideration of extension of the relation of divisibility to ordinals. The left cancellation law on ordinals leads to the concept of left divisibility, the highest common left factor (hclf ($\wedge$)) and dually the least common right multiple (lcrm ($\vee$)) which induces primeness of ordinals and enriches the ordinal number theory. We shall determine the lattice structures of ordinal numbers.

Keywords: Ordinal Number, Ordinal Quotient, Minimal Sequence, Lattice Ordered Binoid, Residuated Lattice.

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Introduction

Cantor indexes the steps of his construction of ordinals by new objects, the transfinite numbers which can be visualized as follows [2, 4, 5, 9, 10, 13, 22, 23]:

$0, 1, 2, \ldots \omega, \omega + 1, \ldots, \omega + \omega \ldots \omega + \omega + \omega, \ldots \omega^2, \ldots \omega^3 \ldots$. But this possesses a problem : how far do we have to go. In fact, Cantor found it very hard to define transfinite number, even using the brand-new-and controversial-vocabulary of set theory.

Neumann, Von J. [20, 21] considered ω as an ordinal of second kind and is the greatest ordinal among ordinals of the first kind. The symbol ε_0 is the least upper bound of the sequence : $1, \omega, \omega^\omega, \omega^{\omega^\omega} \ldots$ and the symbol Ω to represent a well ordered set consisting of all the ordinals of the first and the second kinds. In von Neumann set theory the class of all ordinals does exist, but it is a proper class and thus can't be a member of itself or any other class.

R.M. Robinson [17, 22], P. Bernays [2, 21] and K. Godel [9, 10] introduced an independent theory of ordinals without referring the definition to the concept of order.

R.N. Lal [19] gave a complete extention of ordinals with non-commutative operations. The difficulty what he felt in density problem due to non-commutativity was resolved through the introduction of ordinal continued fraction. His concept of fraction is quite different from that of E. Zakon [25]. He introduced ordinal rationals with their incompleteness and ordinals reals with their completeness in detail.

Only a few Mathematicians have worked on lattices infused with ordinals. S. Kuhlmann [18, 19, 25] discovered some conditions under which chains can be embedded into ordinals. O. Bernard [1] invested regular contexts of ordinal sums and ordinal products of two lattices. In this paper we shall form the lattice structures of ordinals. For terminologies we refer to the Lattice Theory by G. Birkhoff [3].

Ordinal Quotient

Definition 2.1. Let $\alpha \neq 0$ and β be two ordinals .Then we say that α divides β (in symbol, α/β) iff there exists an ordinal γ such that $\beta = \alpha*\gamma$. Here γ is called the left quotient of β by α and is designated as $[\beta : \alpha]$. A non-zero ordinal γ is called :

(I) the hclf of α and β (i.e., $\gamma = \alpha \wedge \beta$) iff γ is the largest left divisor of both α and β, i.e. iff $\gamma \mid \alpha$, $\gamma \mid \beta$ and further $\delta \mid \alpha$ and $\delta \mid \beta$ implies that $\delta \mid \gamma$ for every $\delta \neq 0$.

(II) the lcrm of two non-zero ordinals α and β (i.e., $\gamma = \alpha \vee \beta$) iff γ is the smallest ordinal such that α and β are left divisors of γ, i.e., iff $\alpha \mid \gamma$, $\beta \mid \gamma$ and further $\alpha \mid \delta$ and $\beta \mid \delta$ implies that $\gamma \mid \delta$ for every δ.

In the ensuring study we shall assume α as a nonzero ordinal number involved in $[\beta : \alpha]$ for every ordinal β.

Theorem 2.1. The quotient formation [:] has the following properties :

(I) $[\alpha : \gamma] + [\beta : \gamma] = [\alpha + \beta : \gamma]$

(II) $[\beta : \gamma] * [\alpha : \beta] = [\alpha : \gamma]$

(III) $[2 * \alpha : 2 * \beta] = [\alpha : \beta]$

(IV) $[2 * \alpha : \beta] = [\alpha : \beta]$ provided α is a limit ordinal.

(V) $[\alpha : \alpha] = 1$

(VI) $[\alpha : \beta] = [1 : [\beta : \alpha]]$

(VII) $[\beta : \alpha] * \gamma = [\gamma : [\beta : \alpha]] = [\beta * \gamma : \alpha]$

(VIII) $[[\alpha : \beta] : \gamma] = [\alpha : \beta * \gamma]$

(IX) $[[\alpha : \gamma] : [\beta : \gamma]] = [(\alpha : \beta]$

Proof. (I) Put $p = [\alpha : \gamma]$ and $q = [\beta : \gamma]$

Then $\alpha = \gamma * p$ and $\beta = \gamma * q$

$\Rightarrow \alpha + \beta = \gamma * (p + q) \Rightarrow p + q = [\alpha + \beta : \gamma]$

$\beta = \alpha * \gamma \Leftrightarrow [\beta : \alpha] = \gamma$

(II) Put $p = [\alpha : \beta]$, $q = [[\beta : \gamma]$

Then $\alpha = \beta * p$ and $\beta = \gamma * q$

$\Rightarrow \alpha = \gamma * q * p \Rightarrow [\alpha : \gamma] = q * p$

(III) Put $p = [2 * \alpha : 2 * \beta]$

Then $2 * \alpha = 2 * \beta * p \Rightarrow \alpha = \beta * p$

(IV) The proof is an immediate consequence of the fact that $2*\alpha = \alpha$ when α is a limit ordinal.

(V) Obvious.

(VI) Put $p = [\alpha : \beta]$, $q = [\beta : \alpha]$ and $r = [1 : q]$

Then $1 = q * r$, $\alpha = \beta * p$, $\beta = \alpha * q$

$\Rightarrow \alpha = \alpha * q * p \Rightarrow 1 = q * p \Rightarrow q * r = q * p \Rightarrow p = r$

(VII) Put $[\beta : \alpha] = p$. Then $\beta = \alpha * p$

$\Rightarrow \beta * \gamma = \alpha * p * \gamma = \alpha * (p * \gamma)$

$\Rightarrow p * \gamma = \beta * \gamma * \alpha$

Also, $[\gamma : [\alpha : \beta]] = \xi$ and $[\alpha : \beta] = \eta$

$\Rightarrow \gamma = \eta * \xi \Rightarrow \beta * \gamma = (\beta * \eta) * \xi \Rightarrow \beta * \gamma = \alpha * \xi$

$\Rightarrow \xi = [\beta * \gamma : \alpha]$

(VIII) Put $[\alpha : \beta] : \gamma] = \xi$

Then $[\alpha : \beta] = \gamma * \xi \Rightarrow \alpha = (\beta * \gamma) * \xi \Rightarrow [\alpha : \beta * \gamma] = \xi$

(IX) Put $[\alpha : \gamma] = p$, $[\beta : \gamma] = q$ and $[p : q] = r$

Then $\alpha = \gamma * p$, $\beta = \gamma * q$ and $p = q * r$

$\Rightarrow \gamma * p = \gamma * (q * r) = (\gamma * q) * r = \beta * r \Rightarrow \alpha = \beta * r$

Corollary 2.1. $\alpha \le \beta$ and $\gamma \ne 0$

$\Rightarrow [\alpha : \gamma] \le [\beta : \gamma]$

Proof. There exists β_1 such that $\beta = \alpha + \beta_1$

$\Rightarrow [\alpha : \gamma] + [\beta_1 : \gamma] = [\beta : \gamma]$

$\Rightarrow [\alpha : \gamma] \le [\beta : \gamma]$

Remark 2.1. In view of the result (vi) of the above theorem we may write $[\alpha : \beta] = [\beta : \alpha]^{-1} = 1/[\beta : \alpha]$ and that of (IX), $[\alpha : \gamma] / [\beta : \gamma] = [\alpha : \beta] = \alpha / \beta$.

Theorem 2.2. (I) ω is a divisor of every ordinal of the second kind.

(II) If α and β are ordinals of the second kind and $\alpha \mid \beta$. Then there exists a pair of ordinals ξ and η such that one of them divides the other.

(III) For any two ordinals α and β,

$(\alpha \wedge \beta) * (\alpha \vee \beta) \le (\beta * \alpha) \vee (\alpha * \beta)$

Proof. (I) For, if α is an ordinal number of the second kind, we can obtain an additively indecomposible ordinal $\omega * \xi$ for some ordinal ξ such that $\alpha = \omega * \xi$.

(II) For, $\omega \mid \alpha$ and $\omega \mid \beta \Rightarrow \exists$ ordinals ξ and η such that

$\alpha = \omega * \xi$ and $\beta = \omega * \eta$

Further, $\alpha / \beta \Rightarrow \exists$ an ordinal γ such that $\beta = \alpha * \gamma$

$\Rightarrow \omega * \eta = \omega * \xi * \eta \Rightarrow \eta = \xi * \gamma$

An element α in ω is said to be minimal (α min) in case $0 < \alpha$ and there exists no element β such that $0 < \beta < \alpha$.

(III) for, $\{(\alpha \wedge \beta) * \alpha\} \vee \{(\alpha \wedge \beta) * \beta\} \le ((\beta * \alpha) \vee (\alpha * \beta)$.

The following theorem induces the notion of minimal sequence of ordinals.

Theorem 2.3. If α is not minimal in ω, then there exists an ordinal β such that $\beta * 2 \le \alpha$.

Proof. If α is not minimal, $\exists\ \beta_1 \varepsilon\ \omega$ such that $\beta_1 < \alpha$ and hence their exists a unique ordinal $\beta_2 < \alpha$ with $\alpha = \beta_1 + \beta_2 \ge \beta + \beta = \beta * 2$, where β is defined as the smaller of β_1 and β_2

Definition 2.2. A minimal sequence (α_i) of elements in ω is one containing but one element α_1 which is minimal, or containing a denumerable infinitude of elements such that

$[\alpha_{i+j} : \alpha_i] \ge 2^j (i, j = 1, 2, \ldots)$

Remark 2.2. There exists a minimal sequence.

At this point we fix attention on minimal sequence.

Lemma 2.1. If $\alpha\ \varepsilon\ \omega$. Then $\lim_{i\to\infty}[\alpha_i : \alpha] = \infty$ in the case minimal sequence is infinite.

Proof : We have by the theorem (2.1)(II)

$[\alpha : \alpha_i] = [\alpha_{i+1} : \alpha_i] * [\alpha : \alpha_{i+1}] \ge 2 * [\alpha_i : \alpha_{i+1}]$

By the induction

$[\alpha : \alpha_{i+j}] \ge 2^j * [\alpha : \alpha_{i+1}]$

Putting $i = 1$ and $j = k - 1$

$[\alpha : \alpha_k] \ge 2^{k-1} * [\alpha : \alpha_2]$

Therefore $\exists\ i_0$ such that

$$[\alpha : \alpha_{i_0}] \ge 2^{i_0 - 1}$$

$$\Rightarrow \lim_{i\to\infty}[\alpha : \alpha_i] = \infty$$

Theorem 2.4. Let $\alpha \ne 0$, $\beta \ne 0$ be given.

Then $\lim_{i\to\infty}[\beta : \alpha_i]/[\alpha : \alpha_i]$ exists and is > 0, < ∞ (If (α_i) consists of one element α_1 min, we mean by $\lim_{i\to\infty}$ the value at $i = 1$).

Proof. (I) Let $\alpha = \alpha_1$ be minimal. Then $\alpha_1 \ne 0$ and $\beta = \alpha_1 * [\beta : \alpha_1]$.

Now $[\beta : \alpha_1] \ne 0$, since $\beta \ne 0$ and similarly $\alpha = \alpha_1 * [\alpha : \alpha_1]$ whence $[\alpha, \alpha_1] \ne 0$.

Hence $\lim_{i\to\infty}\dfrac{[\beta : \alpha_i]}{[\alpha : \alpha_i]} = \dfrac{[\beta : \alpha_1]}{[\alpha : \alpha_1]}$ exists and has the desired property.

(II) Let (α_i) be infinite and minimal. Then by Theorem 2.1(II)

$[\beta : \alpha_{i+1}] = [\alpha_i : \alpha_{i+1}] * [\beta : \alpha_i] \le [\alpha_i : \alpha_{i+1}] + 1) * ([\beta : \alpha_i] + 1)$

$[\alpha : \alpha_{i+1}] = [\alpha_i : \alpha_{i+1}] * [\alpha : \alpha_i]$

Therefore,

$$\frac{[\beta : \alpha_{i+1}]}{[\alpha : \alpha_{i+1}]} \le \left(\frac{[\alpha_i : \alpha_{i+1}] + 1}{[\alpha_i : \alpha_{i+1}]}\right) * \left(\frac{[\beta : \alpha_{i+1}] + 1}{[\alpha : \alpha_i]}\right)$$

$$= (1 + 2) * \left(\frac{[\beta : \alpha_i] + 1}{[\alpha : \alpha_i]}\right)$$

$$= 3 * \left(\frac{[\beta : \alpha_i] + 1}{[\alpha : \alpha_i]}\right)$$

$$= \frac{[\beta : \alpha_i] + 1}{[\alpha : \alpha_i]}$$

Therefore, $\lim\limits_{i\to\infty}\dfrac{[\beta:\alpha_{i+j}]}{[\alpha:\alpha_{i+j}]}\le\dfrac{[\beta:\alpha_i]+1}{[\alpha:\alpha_i]}$

That is $\lim\limits_{k\to\infty}\dfrac{[\beta:\alpha_k]}{[\alpha:\alpha_k]}\le\dfrac{[\beta:\alpha_i]+1}{[\alpha:\alpha_i]}$

So that

$$\lim_{k\to\infty}\frac{[\beta:\alpha_k]}{[\alpha:\alpha_k]}\le\lim_{k\to\infty}\frac{[\beta:\alpha_k]+1}{[\alpha:\alpha_k]}=\lim_{i\to\infty}\frac{[\beta:\alpha_i]}{[\alpha:\alpha_i]}$$

Since $\dfrac{1}{[\alpha:\alpha_i]}$ tends to zero by the lemma 2.1. But this implies the existence of desired limit. This is finite also since this is less than or equal to $\dfrac{[\beta:\alpha_i]+1}{[\alpha:\alpha_i]}$ and since this is finite if i is sufficiently great. Since the same reasoning applies to the reciprocal $\dfrac{[\alpha:\alpha_i]}{[\beta:\beta_i]}$, this fraction has also a finite limit. Hence the limit of $\dfrac{[\beta:\alpha_i]}{[\alpha:\alpha_i]}$ exists. This completes the proof.

Remark 2.3. The finite limit obtained above represents an ordinal real number.

Definition 2.3. If $\alpha\ne 0$, $\beta\ne 0$, we define

$$(\beta:\alpha)=\lim_{i\to\infty}[\beta:\alpha_i]/[\alpha:\alpha_i]$$

Theorem 2.5. Let α, β, γ are different from 0. Then the following results hold:

(a) $(\alpha, \alpha) = 1$

(b) $(\alpha:\beta)=(\beta:\alpha)^{-1}$

(c) $(\alpha:\gamma)=(\beta:\gamma)*(\alpha:\beta)$

(d) $(\alpha+\beta:\gamma)=(\alpha:\gamma)+(\beta:\gamma)$

(e) $\alpha>\beta\Rightarrow(\alpha:\gamma)>(\beta:\gamma)$

Proof. (a) This is obvious, since $[\alpha:\alpha_i]/[\alpha:\alpha_i]=1$, for sufficiently large i,

(b)

$$(\alpha:\beta)=\lim_{i\to\infty}[\alpha:\alpha_i]/[\beta:\alpha_i]=\lim_{i\to\infty}\left([\beta:\alpha_i]/[\alpha:\alpha_i]\right)^{-1}=(\beta:\alpha)^{-1}$$

(c)

$$(\alpha:\gamma)=\lim_{i\to\infty}[\alpha:\alpha_i]/[\gamma:\alpha_i]=\lim_{i\to\infty}\left([\alpha:\alpha_i]/[\beta:\alpha_i]\right)*\left(\lim_{i\to\infty}[\beta:\alpha_i]/[\gamma:\alpha_i]\right)=(\beta:\alpha)*(\beta:\gamma)$$

(d) (I) Let α_l be minimal, and put $p=[\alpha:\alpha_1]$, $q=[\beta:\alpha_1]$ and $s=[\gamma:\alpha_1]$.

Then $\alpha=\alpha_1*p$, $\beta=\alpha_1*q$, $\gamma=\alpha_1*s$

$\therefore\ \alpha+\beta=\alpha_1*(p+q)\Rightarrow[\alpha+\beta:\alpha_1]=p+q$

and $(\alpha+\beta:\gamma)=[\alpha+\beta:\alpha_1]/[\gamma:\alpha_1]=p+q/s=p/s+q/s=(\alpha:\gamma)+(\beta:\gamma)$

(II) Let α_i be infinite (i = 1, 2, 3, …). Then by the theorem 2.1(I)

$[\alpha:\alpha_i]+[\beta:\alpha_i]=[\alpha+\beta:\alpha_i]$

If we divide by $[\gamma:\alpha_i]$ and let i tend to infinity, each term has a limit and from which (d) follows:

(e) There exists β_1 with $\alpha=\beta+\beta_1$ and $\beta_1\ne 0$

Then $(\alpha:\gamma)=(\beta:\gamma)+(\beta_1:\gamma)>(\beta:\gamma)$

3. Lattice ordered binoid (lobinoid) [12, 13, 14, 15]

In this section we shall study the lattice ordered structure of the set of nonzero ordinal numbers: α, β, γ, …

Lemma 3.1. The relation of divisibility ($\le$) partially orders the set ω of non zero ordinal numbers with respect to which the monoid (ω, $*$) is left ordered. The poset with hclf ($\wedge$) and lcrm ($\vee$) is a lattice.

Proof. (I) That $\le$ is a porelation and helf and lcrm are lattice operations are obvious.

(II) $\alpha\le\beta$ (i.e. $\alpha\mid\beta$) $\Rightarrow\exists$ an ordinal γ such that $\beta=\alpha*\gamma$

$\Rightarrow\delta*\beta=\delta*(\alpha*\gamma)$

$=(\delta*\alpha)*\gamma=\delta*\beta\Rightarrow\delta*\alpha\,|\,\delta*\beta.$

Theorem 3.1. The set (ω, +, $*$) is a lobinoid.

Proof : (ω, +, $*$) equipped with associative binary operation is a bisemigroup with the property : $n+\omega=n\omega=\omega$ for every ordinal n of the first kind. Thus the existence of a common natural element ω for both the associative binary operations together with the lemma 3.1 yield the proof of the theorem.

Definition 3.1. An ordinal number α is said to be even iff $2\mid\alpha$; odd, otherwise, prime iff for every ordinal number γ either $\alpha\mid\gamma$ or hclf (α, γ) = 1. Two ordinals α and β are said to be co-prime iff hclf (α, β) = 1.

Example 3.1. An initial transfinite ordinal number ω is even whereas $\omega+1$ is odd and they are prime. Further Ω is prime.

Solution. Since $\omega=2*\omega$ and $\omega+1=2*\omega+1$

Remark 3.1. Since every ordinal can be represented either as $2*\alpha$ or as $2*\alpha+1$, each ordinal number is either even or odd, for example

$(\omega+1)*2=2*(\omega+2)+1$ is odd.

The Fermat's factorization method (i.e., an odd number can be factorized iff it is a difference of two squares [6, 14, 15, 16]) fails for ordinals, since

$\omega^{n+1}=\omega^n*\omega=2*\omega^n*\omega$ is an even ordinal and

$\omega^{n+1}=(\omega^n*\omega)^2-(\omega^n)^2$

Again ω^2 has infinitely many such representations: $\omega^2=[(\omega*(n+1)]^2-[\omega*n]^2$, for n = 1, 2, …

Moreover, Goldback hypothesis (i.e., every even natural number > 2 is the sum of two prime numbers [23, p. 21]) is also a sum. We have the following extension of the familiar factorization theorem for finite ordinals :

Theorem 3.2. Every ordinal $\alpha>1$ which is not itself a prime, is a product of a finite number of primes, sometimes in more than one-way.

Proof. There is a least ordinal $\beta>1$ such that $\alpha=\alpha_1*\beta$, where $1<\alpha_1<\alpha$. If α_l is not a prime, the argument may be repeated on α_1. Since a decreasing sequence of ordinals must be finite, we have the desired result.

As an example of non-uniqueness we may cite:

$$\omega^2 = (\omega + 1) * \omega = (\omega + 1) * 2 * \omega$$
$$= (\omega + 1) * 3 * 2 * \omega = 5 * (\omega + 1) * 7 * \omega$$

etc.

One of the most important concept in ω is that of residual. Historically speaking it draws primary inspiration for the work M. Ward and R.P. Dilworth appearing in a series of important papers [6, 7, 8, 16, 23, 24]. Our study is naturally based on non-commutative case. The right residual $\alpha : \beta$ of α by β is the largest γ such that $\beta * \gamma \leq \alpha$. A right residuated lattice is an *l*-groupoid (Lattice-groupoid) in which right residual for every pair of elements exists [3].

Theorem 3.3. ω is right residuated.

Proof : For ordinals α and β in ω there exists a unique ordinal γ and a unique ordinal $\delta < \alpha$ in ω such that

$$\alpha = \beta * \gamma + \delta \Rightarrow \beta * \gamma \leq \alpha$$

Remark 3.2. From the lemma 3.1 and the theorems 3.1, 3.2 and 3.3 it follow that ω is a right residuated lobinoid.

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